AP® CALCULUS AB
2008 SCORING GUIDELINES

Question 3

Oil is leaking from a pipeline on the surface of a lake and forms an oil slick whose volume increases at a constant rate of 2000 cubic centimeters per minute. The oil slick takes the form of a right circular cylinder with both its radius and height changing with time. (Note: The volume V of a right circular cylinder with radius r and height h is given by $V = \pi r^2 h.$)

(a) At the instant when the radius of the oil slick is 100 centimeters and the height is 0.5 centimeter, the radius is increasing at the rate of 2.5 centimeters per minute. At this instant, what is the rate of change of the height of the oil slick with respect to time, in centimeters per minute?

(b) A recovery device arrives on the scene and begins removing oil. The rate at which oil is removed is $R(t) = 400\sqrt{t}$ cubic centimeters per minute, where t is the time in minutes since the device began working. Oil continues to leak at the rate of 2000 cubic centimeters per minute. Find the time t when the oil slick reaches its maximum volume. Justify your answer.

(c) By the time the recovery device began removing oil, 60,000 cubic centimeters of oil had already leaked. Write, but do not evaluate, an expression involving an integral that gives the volume of oil at the time found in part (b).

Solution	Scoring
(a) When $r = 100$ cm and $h = 0.5$ cm, $\frac{dV}{dt} = 2000\ \text{cm}^3/\text{min}$ and $\frac{dr}{dt} = 2.5\ \text{cm}/\text{min}$. $\frac{dV}{dt} = 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}$ $2000 = 2\pi(100)(2.5)(0.5) + \pi(100)^2 \frac{dh}{dt}$ $\frac{dh}{dt} = 0.038$ or 0.039 cm/min	$4 : \begin{cases} 1 : \frac{dV}{dt} = 2000 \text{ and } \frac{dr}{dt} = 2.5 \\ 2 : \text{expression for } \frac{dV}{dt} \\ 1 : \text{answer} \end{cases}$
(b) $\frac{dV}{dt} = 2000 - R(t)$, so $\frac{dV}{dt} = 0$ when $R(t) = 2000$. This occurs when $t = 25$ minutes. Since $\frac{dV}{dt} > 0$ for $0 < t < 25$ and $\frac{dV}{dt} < 0$ for $t > 25$, the oil slick reaches its maximum volume 25 minutes after the device begins working.	$3 : \begin{cases} 1 : R(t) = 2000 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$
(c) The volume of oil, in cm^3, in the slick at time $t = 25$ minutes is given by $60{,}000 + \int_0^{25} (2000 - R(t))\, dt$.	$2 : \begin{cases} 1 : \text{limits and initial condition} \\ 1 : \text{integrand} \end{cases}$

Work for problem 3(a)

$\frac{dV}{dt} = 2000 \frac{cm^3}{min}$ $\frac{dr}{dt} = 2.5 \frac{cm}{min}$ $\frac{dh}{dt} = ?$

$r = 100\ cm$ $h = .5\ cm$

$V = \pi r^2 h$

$\frac{dV}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt} \cdot h + \pi r^2 \cdot \frac{dh}{dt}$

$2000 = \pi\left(2 \cdot 100 \cdot 2.5 \cdot .5 + (100)^2 \cdot \frac{dh}{dt}\right)$

$\frac{2000}{\pi} - 250 = 100^2 \cdot \frac{dh}{dt}$

$\frac{dh}{dt} = .039 \frac{cm}{min}$

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Continue problem 3 on page 9.

3A2

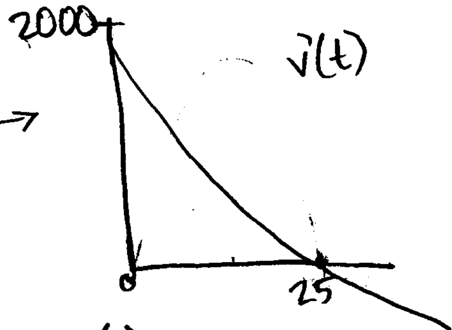
Work for problem 3(b)

$V(t) = \int_0^t 2000 \cdot dt - \int_0^t 400\sqrt{t} \cdot dt$

$V'(t) = 2000 - 400\sqrt{t}$ $\cancel{= 2000t - \frac{800}{3} t^{3/2}}$

$0 = 2000 - 400\sqrt{t}$

$t = 25$ min



*point where v'(t) changes from (+) to (-) is where v(t) has a local maximum

→ $v''(t) = -\frac{200}{\sqrt{t}}$ always negative when $t > 0$ → v(t) always concave down making the local max. of t=25 a global maximum

*t=0 is when device began working

Work for problem 3(c)

$\text{Volume} = \int_0^{25} 2000 - 400\sqrt{t} \cdot dt + 60{,}000$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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3B1

Work for problem 3(a)

$\frac{dV}{dt} = 2000 \text{ cm}^3/\text{min}$

$V = \pi r^2 h$

when $r = 100$ $h = 0.5$

$\frac{dr}{dt} = 2.5$

find: $\frac{dh}{dt}$!

$\frac{dV}{dt} = 2\pi r \frac{dr}{dt} \cdot h + \pi r^2 \cdot \frac{dh}{dt}$

$2000 = 2 \cdot \pi \cdot 100 \cdot 0.5 \cdot 2.5 + \pi \cdot 10000 \cdot \frac{dh}{dt}$

$2000 - 785.398 = 10000\pi \cdot \frac{dh}{dt}$

$1214.602 = 10000\pi \cdot \frac{dh}{dt}$

$0.387 = \frac{dh}{dt}$

$\frac{dh}{dt}$ at the given instant is 0.387 cm/min

↳ the height of the oil slick is changing at that rate

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Continue problem 3 on page 9.

3B2

Work for problem 3(b)

$R(t) = 400\sqrt{t}$ cm^3/min

Volume of Oil present = $\cancel{\int 2000 - 400\sqrt{t}}$

$V = 2000 - 400\sqrt{t}$

$2000 - 400\sqrt{t} = 0$

$t = 25$

after 25 minutes the Volume reaches its maximum value.

Work for problem 3(c)

$V = 6000 - \int_0^{25} (2000 - 400\sqrt{t})\,dt$ = volume of oil at the time $t = 25$

END OF PART A OF SECTION II

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3C1

Work for problem 3(a)

$\frac{dv}{dt} = 2000 \text{cm}^3/\text{min}$ $V = \pi r^2 h$

$r = 100$ $h = .5$ $\frac{dr}{dt} = 2.5 \text{cm}/\text{min}$

$\frac{dv}{dt} = \pi 2r \frac{dr}{dt} \frac{dh}{dt}$

$2000 = 2\pi(100)(2.5)\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{4}{\pi} \text{cm}/\text{min}$ = rate of change of height

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Continue problem 3 on page 9

3C2

Work for problem 3(b)

$\frac{dv}{dt}$ = 2000 cm/min (rate increases by)

2000 - 400$\sqrt{t}$ = 0

@ $t = \sqrt{5}$

~~The oil slick reaches its maximum volume @ t = 0 because the amount of oil being removed is least when t = 0) Since the oil is leaking at a constant rate, the factor that is important is how much is being removed, and that increases as times goes by, as shown by R(t)~~

The oil slick reaches its max volume @ $t = \sqrt{5}$ because after $t = \sqrt{5}$, the recovery device begins pumping more than 2000 cm^3 of oil out of the lake, so the rate at which oil was entering the pond went from increasing to decrease ∴ a max was attained

Work for problem 3(c)

$$\text{Oil} = 60000 + \int_0^{\sqrt{5}} 2000 - R(t)\,dt$$

END OF PART A OF SECTION II

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Question 3

Overview

This problem presented students with a scenario in which oil leaking from a pipeline into a lake organizes itself as a dynamic cylinder whose height and radius change with time. The rate at which oil is leaking into the lake was given as 2000 cubic centimeters per minute. Part (a) was a related-rates problem; students needed to use the chain rule to differentiate volume, $V = \pi r^2 h,$ with respect to time and determine the rate of change of the oil slick's height at an instant when the oil slick has radius 100 cm and height 0.5 cm, and its radius is increasing at $2.5 \text{ cm/min}.$ In part (b) an oil recovery device arrives on the scene; as the pipeline continues to leak at 2000 cubic centimeters per minute, the device removes oil at the rate of $R(t) = 400\sqrt{t}$ cubic centimeters per minute, with t measured in minutes from the time the device began removing oil. Students were asked for the time t when the volume of the oil cylinder is greatest. They needed to recognize the rate of change of the volume of oil in the lake, $\frac{dV}{dt},$ as the difference between the rate at which oil enters the lake from the leak and the rate at which it is removed by the device. A sign analysis of $\frac{dV}{dt}$ or an application of the Second Derivative Test and the critical point theorem could justify that the critical point found yields a maximum value for the volume of the oil cylinder. Part (c) tested students' ability to use the Fundamental Theorem of Calculus to find the amount of oil in the lake at the time found in part (b), given that 60,000 cubic centimeters had already leaked when the recovery device began its task.

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

The student earned 6 points: 3 points in part (a), 2 points in part (b), and 1 point in part (c). In part (a) the student correctly notes that $\frac{dV}{dt} = 2000$ and $\frac{dr}{dt} = 2.5$ and gives the correct symbolic expression for $\frac{dV}{dt}.$ The student earned the first 3 points. The student makes an error in calculating the numerical value for $\frac{dh}{dt}$ so did not earn the last point. In part (b) the student earned the first 2 points for solving $2000 - 400\sqrt{t} = 0$ to find $t = 25$ as the time when the volume reaches a maximum. The student provides no justification and did not earn the third point. In part (c) the student uses the correct initial condition and the correct limits of integration, earning the first point. The integrand is the negative of what is needed to calculate the volume, so the integrand point was not earned.

Question 3 (continued)

Sample: 3C
Score: 4

The student earned 4 points: 1 point in part (a), 1 point in part (b), and 2 points in part (c). In part (a) the student correctly notes that $\frac{dV}{dt} = 2000$ and $\frac{dr}{dt} = 2.5$ but does not use the product rule correctly. The student did not earn either derivative point and was not eligible for the answer point. In part (b) the student solves $2000 - 400\sqrt{t} = 0$ incorrectly, which earned the first point but not the second one. The student did not earn the third point. The justification is not correct for the student's value of t and is only a local argument where a global argument is required. In part (c) the student earned both points. The initial condition is correct, and the student is allowed to import the incorrect value of t found in part (b) as the upper limit of integration. The integrand is correct.