CollegeBoard
Advanced Placement Program

AP® Calculus

A Sampler of Visual Proofs in First-year Calculus

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Introduction

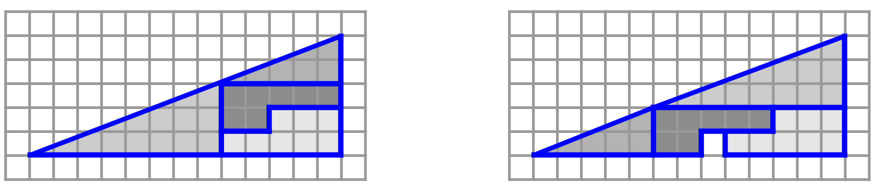
Although neither the noun *proof* nor the verb *prove* appears in the College Board's *Course Description* for Calculus AB and BC [6], I'm sure that theorems and their proofs are an integral part of virtually every calculus course, AP® or otherwise. Hence I will not attempt to convince this audience of the necessary and proper role of proof in calculus—see, for example the articles *Seeing is Believing* by J. T. Sutcliffe [16], *Why We Use Theorem in Calculus* by Lisa Townsley [17], and *Some Thoughts on* 2003 *Calculus AB Question* 6 by Jim Hartman and Larry Riddle [7].

So, what are the characteristics of a *good* proof? I agree with Yuri Manin, who wrote "A good proof is one that makes us wiser" [15], and Andrew Gleason: "Proofs really aren't there to convince you that something is true—they're there to show you *why* it is true" [15].

A good visual proof is a picture or diagram that helps one see *why* a particular mathematical statement is true, and also to see *how* one might begin to prove it true.

Of course, there are lots of bogus visual "proofs" around, too. Here's one from the April 2007 issue of *Math Horizons* [2].

The area A of a triangle equals $A - 1$. "Proof":



(Sometimes I wish this were true, since it would imply that all positive integers are equal. That would certainly make my task of grading exams much easier!)

In this talk, I will discuss several topics from first-year calculus where I've found visual proofs useful.

Better approximations to *e*

Although it's not explicitly on the AP® syllabus, virtually every calculus text (and presumably every calculus course) has the following limit expression for e:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e, \tag{1}$$

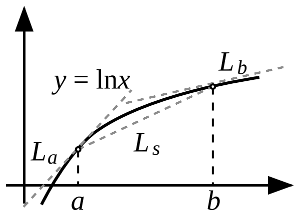
and consequently the approximation for "large" n,

$$\left(1+\frac{1}{n}\right)^n \approx e.$$

This approximation can be easily improved. But first, let's establish (1) using *Napier's inequality*, for which we will give two visual proofs [12]:

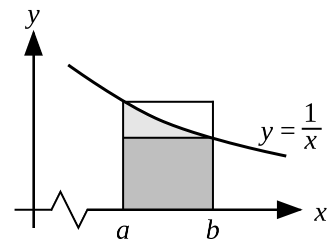
$$0 < a < b \quad \Rightarrow \quad \frac{1}{b} < \frac{\ln b - \ln a}{b-a} < \frac{1}{a}.$$

First semester calculus:



$$m(L_b) < m(L_s) < m(L_a)$$

Second semester calculus:



$$\frac{1}{b}(b-a) < \int_a^b \frac{1}{x}\,dx < \frac{1}{a}(b-a)$$

Now let $a = n$ and $b = n+1$, and apply some simple algebra and the squeeze theorem:

$$\frac{1}{n+1} < \ln\frac{n+1}{n} < \frac{1}{n}$$

$$\frac{n}{n+1} < \ln\left(1+\frac{1}{n}\right)^n < 1$$

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = 1$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

It is also easy to show that $\frac{1}{n+1} < \ln\frac{n+1}{n} < \frac{1}{n}$ implies that

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}. \tag{2}$$

But better approximations to $\int_a^b \frac{1}{x}\,dx$ can dramatically sharpen the inequality:

Taking reciprocals in Napier's inequality yields (for $0 < a < b$)

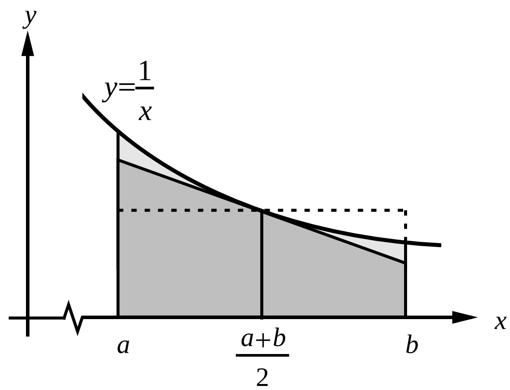
$$a < \frac{b-a}{\ln b - \ln a} < b\,.$$

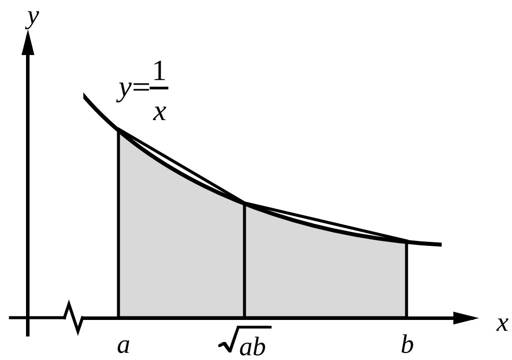
The middle term is called the *logarithmic mean* of a and b. How does the logarithmic mean compare to other means such as the *arithmetic mean* $(a+b)/2$, the *geometric mean* $\sqrt{ab}$, and the *harmonic mean* $2ab/(a+b)$?

Example: The logarithmic mean of 2 and 8 is $3/\ln 2 \approx 4.328$. Here the arithmetic mean is 5, the geometric mean is 4, and the harmonic mean is 3.2; and the logarithmic mean lies between the arithmetic mean and the geometric mean. Is this always true? Yes—

Theorem: $0 < a < b \;\Rightarrow\; \sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}$.

Here is a visual proof using the midpoint rule approximation and a modified trapezoidal rule approximation to the value of the integral [13]:





$$\ln b - \ln a > \frac{2}{a+b}(b-a) \qquad \ln b - \ln a < \frac{b-a}{2\sqrt{ab}} + \frac{b-a}{2\sqrt{ab}}$$

$$\frac{a+b}{2} > \frac{b-a}{\ln b - \ln a} \qquad \sqrt{ab} < \frac{b-a}{\ln b - \ln a}$$

(The point $\sqrt{ab}$ was chosen in the second figure since it minimizes the sum of the areas of the two trapezoids—a nice first-semester calculus problem!)

Combining the above inequalities, taking reciprocals, and setting $a = n$, $b = n+1$ yields

$$\frac{2}{2n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{\sqrt{n(n+1)}},$$

or equivalently

$$\left(1+\frac{1}{n}\right)^{\sqrt{n(n+1)}} < e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}. \qquad (3)$$

Note that the two exponents on $(1+1/n)$ above are the geometric and arithmetic means of n and $n+1$.

So how does the "new" inequality (3) compare to the "traditional" one (2)? Here are some data with $n = 10$ and $n = 50$:

	$n = 10$	$n = 50$
$(1+1/n)^n$	2.59374 (–4.48%)	2.69159 (–0.982%)
$(1+1/n)^{n+1}$	2.85312 (+4.96%)	2.74542 (+0.998%)
$(1+1/n)^{\sqrt{n(n+1)}}$	2.71725 (–0.038%)	2.71824 (–0.0016%)
$(1+1/n)^{n+1/2}$	2.72034 (+0.076%)	2.71837 (+0.0033%)

Note the remarkable improvement in percent relative error (in parentheses)!

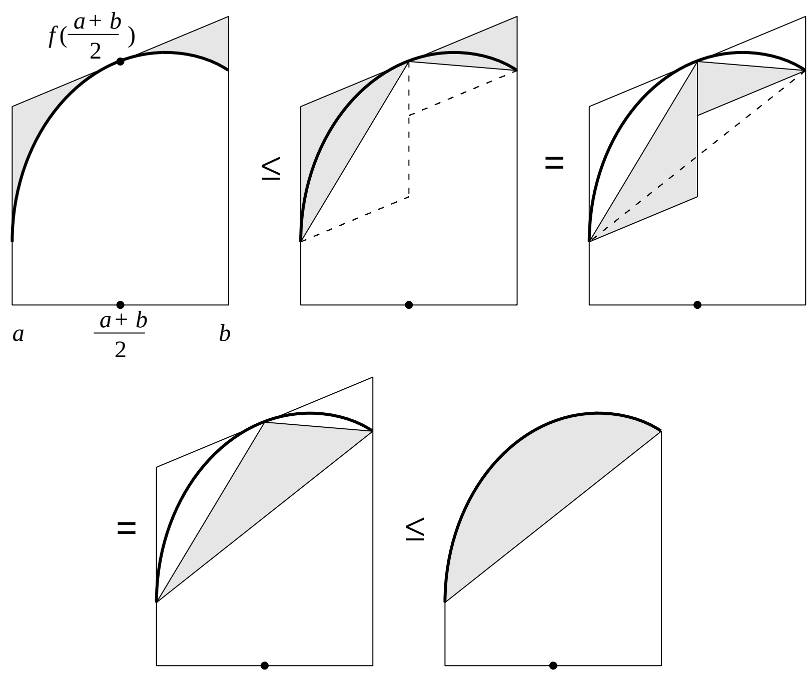
Here the trapezoidal rule outperforms the midpoint rule, but recall that I used twice as many trapezoids—usually the midpoint rule is better.

Recall the usual error estimates for the midpoint and trapezoidal rules:

$$\left|\int_a^b f(x)dx - M_n\right| \le \frac{(b-a)^3}{24n^2} \max_{x\in[a,b]} |f''(x)|;$$

$$\left|\int_a^b f(x)dx - T_n\right| \le \frac{(b-a)^3}{12n^2} \max_{x\in[a,b]} |f''(x)|.$$

Calculus texts often cite the larger denominator as an indication that the midpoint rule is usually more accurate than the trapezoidal rule. However, the “larger denominator” doesn’t justify the claim that the midpoint rule is more accurate (the accuracy also depends on how tight the bounds are). Here’s a visual argument (for concave down functions) [5]:



Question AB/BC 7 *not* on this year’s exam:

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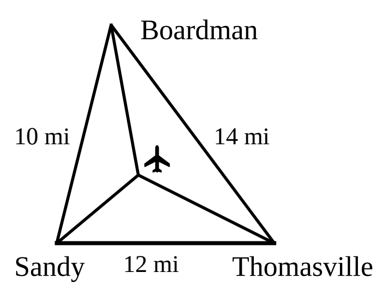
7. Three Oregon towns—Thomasville, Sandy, and Boardman—are situated at the vertices of a triangle, as shown at the right. At some point within the triangle we wish to build a new airport. Find the location of the airport that will minimize the sum of the distances to the three towns.

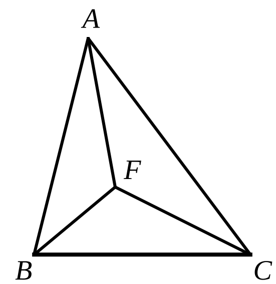
   Justify your answer, and indicate units of measure.



Why is this problem *not* on the exam? Although it is an optimization problem, calculus is *not* the optimal way to solve it!

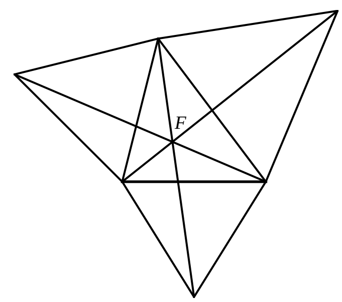
The problem is actually rather old. It is often referred to as *Fermat's problem for Torricelli*. Pierre de Fermat (1601-1665) gave this problem to Evangelista Torricelli (1608-1647), and Torricelli solved the problem in several ways. It is also known as *Steiner's problem* [14]:

Determine a point F in (or on) a given triangle ABC such that the sum $FA + FB + FC$ is a minimum. (The point F is called the *Fermat point* of the triangle.)



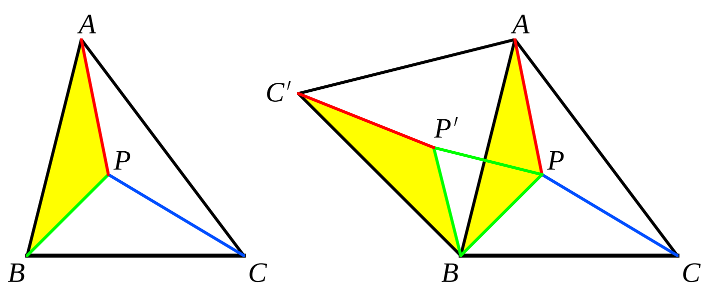
If ABC contains an angle of 120° or more, then the vertex of this angle is the Fermat point. So we will consider only triangles in which each angle measures less than 120°.

There is a simple way to locate the Fermat point of such a triangle. Construct equilateral triangles on the sides of ABC, and join each vertex of ABC to the exterior vertex of the opposite equilateral triangle. Those three lines intersect at the Fermat point!



So why does this work? Our answer is from [14].

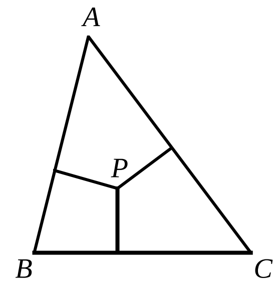
Take any point P inside ABC, connect it to the three vertices, and rotate the triangle ABP (shaded in the figure) 60° counterclockwise as shown to triangle $C'BP'$, and join P' to P:



Notice that $AP = C'P'$ and $BP = P'P$ since triangle $BP'P$ is equilateral, and so $AP + BP + CP = C'P' + P'P + PC$. So the sum $AP + BP + CP$ will be a minimum when P and P' lie on the straight line joining C' to C. There is nothing special about side AB and the new vertex C'—we could equally well have rotated BC or AC counterclockwise (or clockwise) about a vertex. Consequently P must also lie on $B'B$ and $A'A$ (not drawn in the figure), and the Fermat point F is P. In addition, each of the six angles at F measures 60°.

This last fact yields a simple way to locate the Fermat point of a triangle ABC [1]. Prepare two transparencies, one with triangle ABC, the second with three line segments emanating from a central point F at 120° angles. When the second transparency is placed on top of the first so that the line segments pass through the vertices of *ABC*, *F* will be at the Fermat point.

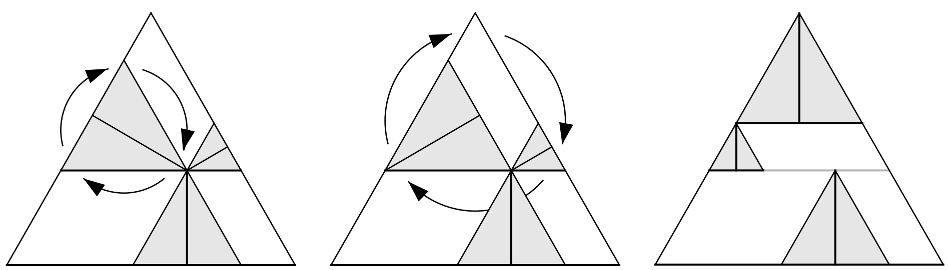
Here is a related problem: Locate the point *P* inside or on the boundary of a triangle so that the sum of the lengths of the perpendiculars to the sides is a minimum.



I'll leave the solution as an exercise (again non-calculus!)—but here is the answer:

1. If the triangle is not equilateral, then *P* is located at the vertex of the largest angle. If there are two largest angles, then there are two locations for *P*.

2. If the triangle is equilateral, then *P* can be located *anywhere* inside or on the boundary of the triangle, since the sum of the perpendicular distances to the sides is constant! This somewhat surprising result is known as *Viviani's Theorem* (Vincenzo Viviani, 1622-1703). Here is a visual proof of Viviani's theorem [8]:



Now back to calculus—

Does 2 = 4?
Well of course not. But consider the following calculations [4]:

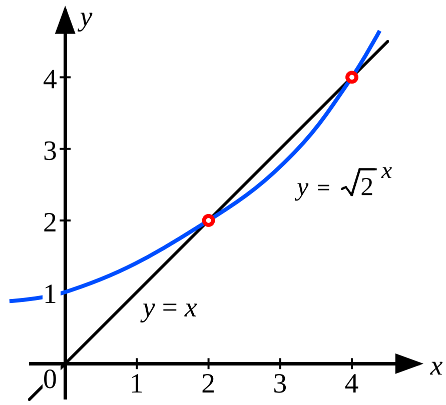
$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2 \Rightarrow x^2 = 2 \Rightarrow x = \sqrt{2}\text{, so } \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}} = 2.$$

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 4 \Rightarrow x^4 = 4 \Rightarrow x = \sqrt{2}\text{, so } \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}} = 4. \ \therefore\ 2 = 4.$$

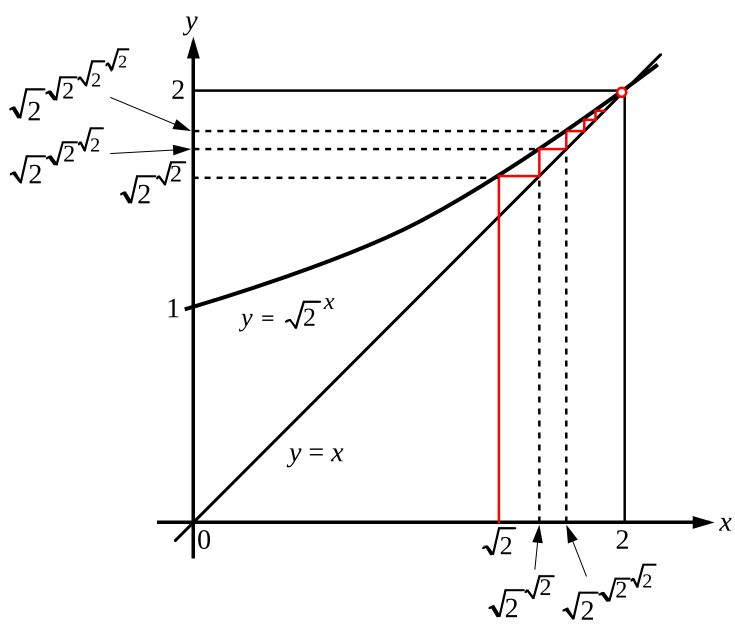
Which calculation (if either) is correct? The answer requires a *definition* of $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}$.

We'll define it as the limit of the sequence $\{\sqrt{2}, \sqrt{2}^{\sqrt{2}}, \sqrt{2}^{\sqrt{2}^{\sqrt{2}}}, \cdots\}$, or equivalently, the limit *a* of the sequence $\{a_1, a_2, a_3, \cdots\}$ with $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2}^{a_n}$ for $n \geq 1$ (the limit exists since the sequence is bounded and monotone).

Taking limits, we have $a = \sqrt{2}^{a}$, so that a is a *fixed point* of the function $f(x) = \sqrt{2}^{x}$. But there are *two* fixed points, namely 2 and 4:



But recalling that the sequence begins with $a_1 = \sqrt{2}$ leads to this visual proof [3] that $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}} = 2$:



Exercises: What about $\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$? $2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}$?

Iteration, fixed points, and geometric series

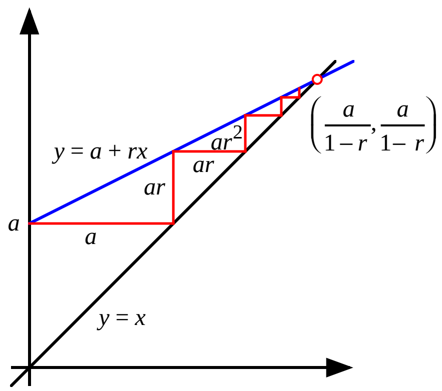
The same idea can be applied to series. The sequence of partial sums for the geometric series $a + ar + ar^2 + \cdots$ is given by

$$s_1 = a,$$
$$s_2 = a + ar,$$
$$s_3 = a + ar + ar^2, \cdots.$$

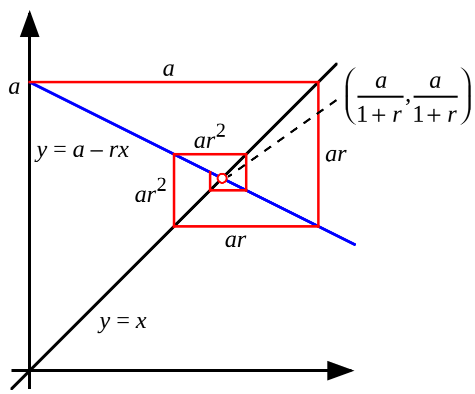
The usual recurrence for this sequence is $s_{n+1} = s_n + ar^n$, but another one is $s_{n+1} = a + rs_n$. Together these two expressions for s_{n+1} immediately imply that $s_n = a(1-r^n)/(1-r)$ for $r \neq 1$ (clearly $s_n = na$ when $r = 1$).

If the series converges to s, then $s = a + rs$; that is, $s = a/(1-r)$ is a fixed point of the linear function $f(x) = a + rx$. This leads to the following visual proof [18] that

$$a + ar + ar^2 + \cdots = \frac{a}{1-r} \text{ for } |r| < 1.$$

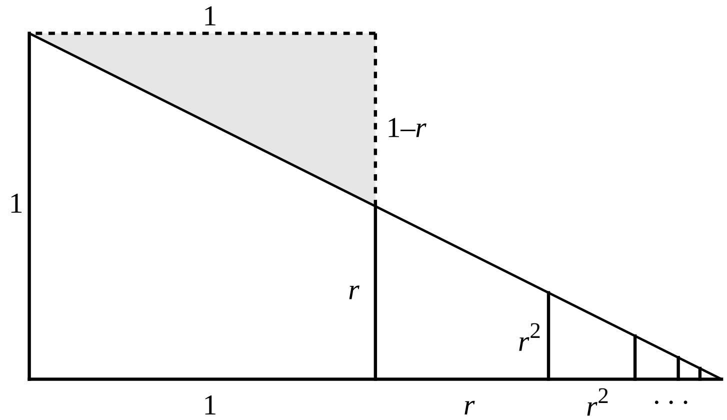


$$a + ar + ar^2 + \cdots = \frac{a}{1-r}, \quad 0 < r < 1$$



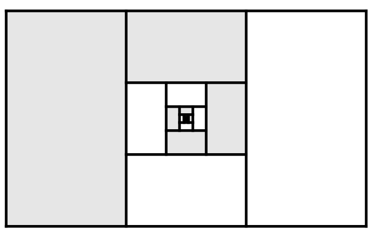
$$a - ar + ar^2 - \cdots = \frac{a}{1+r}, \quad 0 < r < 1$$

Another approach is to *use geometry* [9] to sum a geometric series! So for $0 < r < 1$:

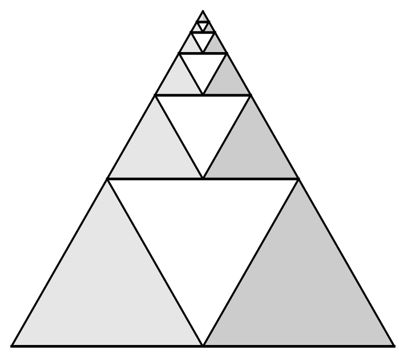


$$\frac{1+r+r^2+\cdots}{1}=\frac{1}{1-r}.$$

And here are a couple of visual proofs for geometric series that employ *self-similarity* [10,11]:



$$\frac{1}{3}+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3+\cdots=\frac{1}{2}$$



$$\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\cdots=\frac{1}{3}$$

I'll conclude with an "application" of a geometric series with $r = 1/2$:

"Theorem": 1 = 2.
"*Proof*":

Let $c_{ij} = \begin{cases} 1, & j=i, \\ -(2^i-1)/2^i, & j=i+1, \\ 0, & \text{otherwise;} \end{cases}$ and set $S=\sum_1^\infty \sum_1^\infty c_{ij}$;

that is, S is the sum of the entries in the matrix

$$\left[c_{ij}\right]=\begin{bmatrix} 1 & -1/2 & 0 & 0 & 0 & \cdots \\ 0 & 1 & -3/4 & 0 & 0 & \cdots \\ 0 & 0 & 1 & -7/8 & 0 & \cdots \\ 0 & 0 & 0 & 1 & -15/16 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The sum of the row sums is $S = 1/2 + 1/4 + 1/8 + \cdots = 1$; the sum of the column sums is $S = 1 + 1/2 + 1/4 + \cdots = 2$; and thus 1 = 2. Where's the error??

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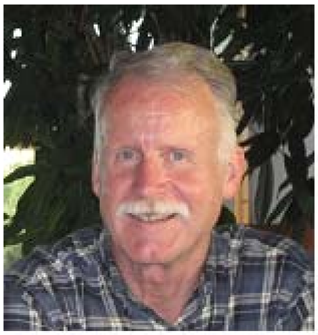

Roger Nelsen grew up in Indiana, and received his B.A. degree in mathematics from DePauw University. He obtained his Ph.D. at Duke University, and since 1969 he has taught at Lewis & Clark College, where he is Professor of Mathematics.

He also has held visiting positions at Mount Holyoke College and the University of Massachusetts at Amherst. Nelsen's research interests lie in the area of mathematical statistics, where he uses copulas (multivariate distribution functions with uniform margins) to study dependence among random variables and to construct families of multivariate distributions. He also is interested in the process of visualization in mathematics, specifically how mathematical drawings help students understand mathematical ideas, proofs, and arguments.

Nelsen has published over 100 research and expository papers and four books: Proofs Without Words: Exercises in Visual Thinking *(MAA, 1993);* Proofs Without Words II: More Exercises in Visual Thinking *(MAA, 2000);* An Introduction to Copulas *(Springer, 1999, 2006); and* Math Made Visual: Creating Images for Understanding Mathematics *(with co-author Claudi Alsina, MAA 2006).*