Question 5

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$. (Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

(a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

(b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

(c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

(d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)\,dt$? Give a reason for your answer.

(a) $r(5.4) \approx r(5) + r'(5)\Delta t = 30 + 2(0.4) = 30.8$ ft Since the graph of r is concave down on the interval $5 < t < 5.4$, this estimate is greater than $r(5.4)$.	2 : { 1 : estimate 1 : conclusion with reason
(b) $\frac{dV}{dt} = 3\left(\frac{4}{3}\right)\pi r^2 \frac{dr}{dt}$ $\left.\frac{dV}{dt}\right\vert_{t=5} = 4\pi(30)^2\, 2 = 7200\pi$ ft^3/min	3 : { 2 : $\frac{dV}{dt}$ 1 : answer
(c) $\int_0^{12} r'(t)\,dt \approx 2(4.0) + 3(2.0) + 2(1.2) + 4(0.6) + 1(0.5)$ $= 19.3$ ft $\int_0^{12} r'(t)\,dt$ is the change in the radius, in feet, from $t = 0$ to $t = 12$ minutes.	2 : { 1 : approximation 1 : explanation
(d) Since r is concave down, r' is decreasing on $0 < t < 12$. Therefore, this approximation, 19.3 ft, is less than $\int_0^{12} r'(t)\,dt$.	1 : conclusion with reason
Units of ft^3/min in part (b) and ft in part (c)	1 : units in (b) and (c)

NO CALCULATOR ALLOWED

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Work for problem 5(a)

$r(5)=30$

$y-30 = 2\cdot(x-5)$

$y-30 = 2\cdot(5.4-5)$

$y-30 = .8$

$\boxed{y=30.8 \text{ ft}}$

this estimate is greater than the actual Value because the graph of $r(t)$ is concave down

Work for problem 5(b)

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{\cancel{3}}\cdot\cancel{3}\pi r^2 \frac{dr}{dt}$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\frac{dV}{dt} = 4\pi \cdot 30^2 \cdot 2$

$\frac{dV}{dt} = 4\pi \cdot 900 \cdot 2$

$\frac{dV}{dt} = 7200\pi \text{ ft}^3/\text{minute}$

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5A 2

NO CALCULATOR ALLOWED

Work for problem 5(c)

$$\int_0^{12} r'(t)\,dt = 2\cdot4 + 3\cdot2 + 2\cdot1.2 + 4\cdot.6 + 1\cdot.5$$
$$= 8+6+2.4+2.4+.5$$
$$= 8+6+4.8+.5$$
$$= 14+5.3$$
$$= \boxed{19.3 \text{ feet}}$$

this is the change in the radius of the baloon from $t=0$ min to $t=12$ min

Work for problem 5(d)

the estimation is less than the actual value because $r'(t)$ is decreasing on the interval $0<t<12$

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5B

NO CALCULATOR ALLOWED

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

Work for problem 5(a)

$$L_n = f(a) + f'(a)(x-a)$$
$$= 30 + 2(x-5) = 30 + 2x - 10 = 20 + 2x$$

Work for problem 5(b)

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \cancel{\frac{3}{3}}\, 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi(30)^2(2) = 4\pi(900)(2) = \boxed{7200\pi \ \frac{\text{feet}^3}{\text{minute}}}$$

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NO CALCULATOR ALLOWED

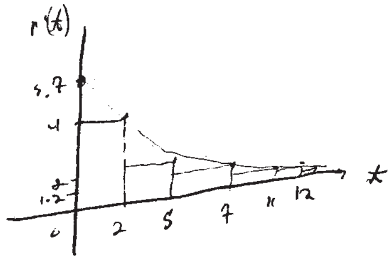
Work for problem 5(c)

RRAM= $1(.5) + \cancel{8}\,4(.6) + 2(1.2) + 3(2) + 4(2) =$

$= .5 + 2.4 + 2.4 + 6 + 8 = \boxed{19.3 \text{ feet}}$

$\int_0^{12} r'(t)dt$ is the sum of the area under the curve of $r'(t)$. It shows the radius of the balloon in feet ~~at~~ 12 minutes after it ~~go~~ begins to expand.

Work for problem 5(d)



The approximation in part c is less than $\int_0^{12} r'(t)dt$ because the ~~areas of a~~ approximated areas within each subinterval fall below the graph of $r'(t)$ because $r'(t)$ is a decreasing function.

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NO CALCULATOR ALLOWED

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Work for problem 5(a)

$R = 30 + 2\times(t-5)$

$R = 30 + 2.5(t-5)$

$R = 30 + 10(t-5)$

$R = 10t - 20$

$R = 10(5.4) - 20$

$R = 34$ feet

This is less than the true value because $r'(t)$ is (+), so the radius is increasing. If r is measured at $t=5$, the measurement will be less than r when $r = 5.4$

Work for problem 5(b)

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 dr$

$\frac{dV}{dt} = 4\pi(30)^2(2)$

$\frac{dV}{dt} = 7200\pi \text{ ft}^3/\text{min}$

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Work for problem 5(c)

$S = 2(4+2+1.2+0.6+0.5)$

$S = 16.6$ feet

$\int_0^{12} r'(t)\,dt = r(t) =$ the radius of the balloon at time t.

Work for problem 5(d)

part (c) approximation is greater than $\int_0^{12} r'(t)\,dt$. Since the radius is increasing, the right value of each approximation is the highest value for each interval, so the approximation is greater.

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Question 5

Overview

The problem presented students with a table of values for the rate of change of the radius of an expanding spherical balloon over a time interval of 12 minutes. Students were told that the radius was modeled by a twice-differentiable function whose graph was concave down. Part (a) asked students to use a tangent line approximation to estimate the radius of the balloon at a specific time and to determine if the estimate was greater than or less than the true value. This tested their ability to use the information about the concavity of the graph of the radius to make the appropriate conclusion about the behavior of the tangent line. In part (b) students had to handle the related rate of change of the volume, given information about the rate of change of the radius. In part (c) students had to recognize the definite integral as the total change, in feet, of the radius of the balloon from time $t = 0$ minutes to time $t = 12$ minutes and approximate the value of this integral using a right Riemann sum and the data in the table. Part (d) asked students to decide if this approximation was greater than or less than the true value of the definite integral. Again, they were required to use the information about the concavity of the graph of the radius to make the appropriate conclusion about the behavior of the graph of the derivative. Units of measure were important in parts (b) and (c).

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

The student earned 6 points: 3 points in part (b), 1 point in part (c), 1 point in part (d), and the units point. In part (a) the student finds the tangent line approximation correctly but does not compute the estimate nor state a conclusion or reason. Thus no points were earned. In part (b) the student correctly finds $\frac{dV}{dt}$ using the chain rule and is therefore eligible for the answer point. This answer is also correct. In part (c) the student finds the correct approximation using a right Riemann sum but fails to provide a correct explanation—this integral represents the *change* in radius, not the radius, after 12 minutes. In part (d) the student correctly identifies the reason that the approximation is less than the actual value: $r'(t)$ is decreasing. The student earned the units point.

Sample: 5C
Score: 4

The student earned 4 points: 3 points in part (b) and the units point. In part (a) the student calculates the estimate for $r(5.4)$ incorrectly and states that $r'(t)$ is positive rather than decreasing. In part (b) the student earned all 3 points for correct use of the chain rule and the correct calculation of the result. In part (c) the student makes an error in calculating the right Riemann sum and does not refer to the *change* in radius. In part (d) the student states that the approximation is greater than the definite integral rather than less. The student earned the units point.