AP® CALCULUS AB
2007 SCORING GUIDELINES

Question 3

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	– 4	4	2
4	–1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

(c) Let w be the function given by $w(x) = \int_1^{g(x)} f(t)\,dt$. Find the value of $w'(3)$.

(d) If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

(a) $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$
$h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$
Since $h(3) < -5 < h(1)$ and h is continuous, by the Intermediate Value Theorem, there exists a value r, $1 < r < 3$, such that $h(r) = -5$.

$2: \begin{cases} 1: h(1) \text{ and } h(3) \\ 1: \text{conclusion, using IVT} \end{cases}$

(b) $\dfrac{h(3) - h(1)}{3 - 1} = \dfrac{-7 - 3}{3 - 1} = -5$
Since h is continuous and differentiable, by the Mean Value Theorem, there exists a value c, $1 < c < 3$, such that $h'(c) = -5$.

$2: \begin{cases} 1: \dfrac{h(3) - h(1)}{3 - 1} \\ 1: \text{conclusion, using MVT} \end{cases}$

(c) $w'(3) = f(g(3)) \cdot g'(3) = f(4) \cdot 2 = -2$

$2: \begin{cases} 1: \text{apply chain rule} \\ 1: \text{answer} \end{cases}$

(d) $g(1) = 2$, so $g^{-1}(2) = 1$.

$$\left(g^{-1}\right)'(2) = \frac{1}{g'\left(g^{-1}(2)\right)} = \frac{1}{g'(1)} = \frac{1}{5}$$

An equation of the tangent line is $y - 1 = \frac{1}{5}(x - 2)$.

$3: \begin{cases} 1: g^{-1}(2) \\ 1: \left(g^{-1}\right)'(2) \\ 1: \text{tangent line equation} \end{cases}$

3A₁

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	−4	4	2
4	−1	3	6	7

Work for problem 3(a)

Differentiability implies continuity so f and g are also continuos for all real numbers. Because f and g are continuous, $h(x)$ is also continuous for all real number. Because h is continuous and $h(3) = -7$ and $h(1) = 3$, $-7 = h(3) < -5 = h(r) < 3 = h(1)$, so a value of r where $h(r) = -5$ is guaranteed by the Intermediate Value Theorem.

Work for problem 3(b)

$h(x)$ continuous and differentable on all real numbers because g and f are continuous and differentiable for all real numbers.

Because $h(x)$ is continuous on $[1,3]$ and differentable on $(1,3)$ and $\frac{h(3)-h(1)}{3-1} = \frac{-7-3}{2} = -5$, the Mean Value Theorem guarantees that there is a value of c where $h'(c) = -5$

Continue problem 3 on page 9.

$3A_2$

Work for problem 3(c)

$u = g(x)$

$u' = g'(x)$

$w'(x) = g'(x) \cdot \dfrac{d}{du} \displaystyle\int_1^u f(t)\,dt$

by the 2^{nd} Fundamental theorem

$w'(x) = g'(x) \cdot f(u) = g'(x) \cdot f(g(x))$

$w'(3) = g'(3) \cdot f(g(3))$

$w'(3) = 2 \cdot f(4)$

$w'(3) = -2$

Work for problem 3(d)

$\dfrac{d}{dx}\left(g^{-1}(2)\right) = \dfrac{1}{g'(1)} = \dfrac{1}{5} = m$

$1 = g(x) \quad x = 2$

$g^{-1}(2) = 1$

$y - 1 = \dfrac{1}{5}(x - 2)$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	–4	4	2
4	–1	3	6	7

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Work for problem 3(a)

$h(1) = f(g(1)) - 6$
$= f(2) - 6$
$= 9 - 6$
$= 3$

$h(3) = f(g(3)) - 6$
$= f(4) - 6$
$= -1 - 6$
$= -7$

according to Rolle's Theorem, if $h(a) = c$ and $h(b) = d$ and g is on the interval $c < g < d$, then there must be a value r that exists on the interval $a < r < b$ for which $h(r) = g$

$h(1) = 3$ and $h(3) = -7$. -5 is on the interval $-7 < -5 < 3$, $\therefore$ there must be a value r for which $h(r) = -5$.

Work for problem 3(b)

$\dfrac{h(3) - h(1)}{3-1} = \dfrac{-7-3}{2} = -5$

according to the Mean Value Theorem, if $\dfrac{F(b) - F(a)}{b-a} = m$, then there must be a value c on the interval $a < c < b$ for which $f(c) = m$

$\dfrac{h(3) - h(1)}{3-1} = -5$ $\therefore$ there must be a value c on the interval $1 < c < 3$ for which $h'(c) = -5$.

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Continue problem 3 on page 9.

 $3B_2$

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Work for problem 3(c)

$w'(x) = g'(x)\,(f(g(x)))$

$w'(3) = g'(3)(f(g(3)))$

$= g'(3) f(4)$

$= (2)(-1)$

$w'(3) = -2$

Work for problem 3(d)

$g(x)$: $y = 2$

$x = 1$

$g^{-1}(x)$ $x = 2$

$y = 1$

$m = 1$

$1 = (1)(2) + c$

$-1 = c$

$y = x - 1$

END OF PART A OF SECTION II

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 3C1

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	−4	4	2
4	−1	3	6	7

Work for problem 3(a)

$h(1) = 3$ and $h(3) = -7$ and since $h'(x)$ is strictly decreasing over this interval, there must be some value, r, over the interval $(1, 3)$ where $h(x) = -5$.

Work for problem 3(b)

$h'(x) = f'(g(x))(g'(x))$

Then, since $f'(x)$ is negative over this interval, the slope of $h'(x)$ must therefore be negative at some point.

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Continue problem 3 on page 9.

 3C₂

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Work for problem 3(c)

$w'(x) = f(g(x))\, g'(x)$

$f(g(3)) \times g'(3)$

-1×2

$\boxed{-2}$

Work for problem 3(d)

$y = \frac{1}{5}x + b$

$3 = \frac{1}{5}(2) + b$

$3 = \frac{2}{5} + b$

$\frac{5}{15} - \frac{6}{15} = b$

$\frac{-1}{15} = b$

$y = \frac{1}{5}x - \frac{1}{15}$

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END OF PART A OF SECTION II

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Question 3

Overview

This problem presented students with a table of selected values of functions f and g, and their first derivatives. A third function h was defined in terms of the composition of f and g. Parts (a) and (b) assessed students' abilities to use the chain rule, the Intermediate Value Theorem, and the Mean Value Theorem to explain why there must be values r and c in the interval $(1, 3)$ where $h(r) = -5$ and $h'(c) = -5$. In part (c) students were given a function w defined in terms of a definite integral of f where the upper limit was $g(x)$. They had to use the Fundamental Theorem of Calculus and the chain rule to calculate the value of $w'(3)$. Part (d) asked students to write an equation for a line tangent to the graph of the inverse function of g at a given value of x. In all parts of this problem students had to use appropriate values from the given table to do their calculations.

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

The student earned 6 points: 1 point in part (a), 2 points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the student correctly identifies $h(1)$ and $h(3)$ for the first point but mistakenly identifies Rolle's Theorem and thus did not earn the second point. In part (b) the student calculates the difference quotient and applies the Mean Value Theorem to support the conclusion and earned both points. In part (c) the student earned both points by correctly applying the Fundamental Theorem of Calculus and the chain rule and by correctly evaluating the function. In part (d) the student correctly declares $g^{-1}(2)$ and earned the first point. The student did not earn the second point for $\left(g^{-1}\right)'(2)$, nor did the student declare a value for $\left(g^{-1}\right)'(2)$, so the student was not eligible for the third point.

Sample: 3C
Score: 4

The student earned 4 points: 1 point in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the student correctly identifies $h(1)$ and $h(3)$ for the first point but does not apply the hypotheses of the Intermediate Value Theorem to support the conclusion and did not earn the second point. In part (b) the student does not calculate the difference quotient and was not eligible for either point. In part (c) the student earned both points by correctly applying the Fundamental Theorem of Calculus and the chain rule and by correctly evaluating the function. In part (d) the student did not earn the first point for $g^{-1}(2)$. The student correctly uses $\left(g^{-1}\right)'(2)$ in the tangent line equation and earned the second point. The student does not declare a value for $g^{-1}(2)$, so was not eligible for the third point.