AP® CALCULUS AB
2007 SCORING GUIDELINES (Form B)

Question 6

Let f be a twice-differentiable function such that $f(2) = 5$ and $f(5) = 2$. Let g be the function given by $g(x) = f(f(x))$.

(a) Explain why there must be a value c for $2 < c < 5$ such that $f'(c) = -1$.

(b) Show that $g'(2) = g'(5)$. Use this result to explain why there must be a value k for $2 < k < 5$ such that $g''(k) = 0$.

(c) Show that if $f''(x) = 0$ for all x, then the graph of g does not have a point of inflection.

(d) Let $h(x) = f(x) - x$. Explain why there must be a value r for $2 < r < 5$ such that $h(r) = 0$.

Solution	Scoring
(a) The Mean Value Theorem guarantees that there is a value c, with $2 < c < 5$, so that $f'(c) = \dfrac{f(5) - f(2)}{5 - 2} = \dfrac{2 - 5}{5 - 2} = -1$.	$2 : \begin{cases} 1 : \dfrac{f(5) - f(2)}{5 - 2} \\ 1 : \text{conclusion, using MVT} \end{cases}$
(b) $g'(x) = f'(f(x)) \cdot f'(x)$ $g'(2) = f'(f(2)) \cdot f'(2) = f'(5) \cdot f'(2)$ $g'(5) = f'(f(5)) \cdot f'(5) = f'(2) \cdot f'(5)$ Thus, $g'(2) = g'(5)$. Since f is twice-differentiable, g' is differentiable everywhere, so the Mean Value Theorem applied to g' on $[2, 5]$ guarantees there is a value k, with $2 < k < 5$, such that $g''(k) = \dfrac{g'(5) - g'(2)}{5 - 2} = 0$.	$3 : \begin{cases} 1 : g'(x) \\ 1 : g'(2) = f'(5) \cdot f'(2) = g'(5) \\ 1 : \text{uses MVT with } g' \end{cases}$
(c) $g''(x) = f''(f(x)) \cdot f'(x) \cdot f'(x) + f'(f(x)) \cdot f''(x)$ If $f''(x) = 0$ for all x, then $g''(x) = 0 \cdot f'(x) \cdot f'(x) + f'(f(x)) \cdot 0 = 0$ for all x. Thus, there is no x-value at which $g''(x)$ changes sign, so the graph of g has no inflection points. OR If $f''(x) = 0$ for all x, then f is linear, so $g = f \circ f$ is linear and the graph of g has no inflection points.	$2 : \begin{cases} 1 : \text{considers } g'' \\ 1 : g''(x) = 0 \text{ for all } x \end{cases}$ OR $2 : \begin{cases} 1 : f \text{ is linear} \\ 1 : g \text{ is linear} \end{cases}$
(d) Let $h(x) = f(x) - x$. $h(2) = f(2) - 2 = 5 - 2 = 3$ $h(5) = f(5) - 5 = 2 - 5 = -3$ Since $h(2) > 0 > h(5)$, the Intermediate Value Theorem guarantees that there is a value r, with $2 < r < 5$, such that $h(r) = 0$.	$2 : \begin{cases} 1 : h(2) \text{ and } h(5) \\ 1 : \text{conclusion, using IVT} \end{cases}$

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Work for problem 6(a)

$f'(c) = -1$ interval $= (2,5)$

$\frac{f(5) - f(2)}{5-2} = \frac{2-5}{3} = -1$

$f'(c) = -1$

According to the Mean Value Theorem, there must exist some c, such that $f'(c) = -1$

Work for problem 6(b)

$g(x) = f(f(x))$

$g'(x) = f'(f(x))f'(x)$ ← Chain Rule

$g'(2) = f'(f(2))f'(2)$

$g'(2) = f'(5)f'(2)$

$g'(5) = f'(f(5))f'(5)$

$g'(5) = f'(2)f'(5$

$f'(5)f'(2) = f'(2)f'(5)$

$\therefore g'(2) = g'(5)$

Mean Value Theorem: $\frac{g'(5) - g'(2)}{5-2} = \frac{0}{3} = 0$, therefore, there must exist some k within $2 < k < 5$ where $g''(k) = 0$

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Work for problem 6(c)

$f''(x)=0$ for all x

pt of inflection on g is where $g''=0$

$g''(x) = f''(f'(f(x)))f'(x) + f'(f(x))f''(x)$

If $f''=0$, then

$g''(x) = 0 + 0$

$= 0$, for every x, meaning there is no point on g where the graph changes concavity.

Work for problem 6(d)

$h(x) = f(x) - x$

$(2,5)$

$h(2) = f(2) - 2$

$= 5 - 2 = \underline{\underline{3}}$

$h(5) = f(5) - 5$

$= 2 - 5 = \underline{\underline{-3}}$

Because the values have opposite signs, according the Intermediate Value Theorem, there must exist some number r such that $h(r) = 0$

The function is continuous ("twice-differentiable") and because it has coordinates above and below the x-axis, there must exist some r.

AB
6B1

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Work for problem 6(a)

$\because f(2) = 5 \quad f(5) = 2$

$\therefore f'(c) = \frac{f(5) - f(2)}{5-2}$

$= \frac{2-5}{5-2} = -1$ (Mean Value Therom)

Work for problem 6(b)

$g'(x) = f'(f(x)) \cdot f'(x)$

$g'(2) = f'(f(2)) \cdot f'(2)$

$g'(5) = f'(f(5)) \cdot f'(5)$

$f(2) = 5 \quad f(5) = 2$

$g'(2) = f'(5) \cdot f'(2)$

$g'(5) = f'(2) \cdot f'(5)$

$\therefore g'(2) = g'(5)$

$\begin{cases} g' \text{ is differentiable on interval } [2,5] \\ g'(2) = g'(5) \\ g' \text{ is contineous.} \end{cases}$

$\therefore$ there is a value k for $2<k<5$ such that $g''(k) = 0$

Continue problem 6 on page 15.

AB
6B[2]

NO CALCULATOR ALLOWED

Work for problem 6(c)

$g' = f'(f(x)) \cdot f'(x)$ $f''(x) = 0$

$g'' = f'(x) \cdot f''(f(x)) \cdot f'(x) + f''(x)\, f'(f(x))$

$= (f'(x))^2 \cdot f''(f(x))$

$\because$ g'' not equal to zero

$\therefore$ the graph of g does not have a point of inflection

Work for problem 6(d)

$h(2) = f(2) - 2 = 3$

$h(5) = f(5) - 2 = 3$

and h is differentiable on $[2, 5]$

therefore, there must be a value r for $2 < r < 5$

such that $h(r) = 0$

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Work for problem 6(a)

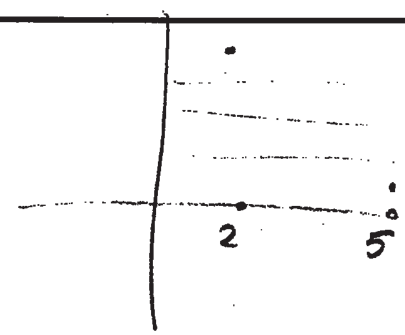
$$\frac{f(5)-f(2)}{5-2} = \frac{2-5}{3} = \frac{-3}{3} = -1$$

because the function is twice differentiable and from the mean value therom

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$= \frac{f(5)-f(2)}{5-2} = \frac{2-5}{3} = \frac{-3}{3} = -1 = f'(c)$$

Work for problem 6(b)



$g(2) = f(f(2))$

$g(2) = f(5) = g(2) = 2$

$g(5) = f(f(5))$

because the function is one to one function that means that the function is either decrease or increase between $(2, 5)$, and it should concave up or down and f is twice differentiable.

Continue problem 6 on page 15.

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Work for problem 6(c)

$f''(x) = 0$
that means the graph doesn't change in concavity, (second derivative is constant), inflection points might be found only when $f''(x)$ changes its sign.

Work for problem 6(d)

$h(x) = f(x) - x$

$h(5) = f(5) - 5 = 2 - 5 = -3$

$h(2) = f(2) - 2 = 5 - 2 = 3$

from Rolle's Therom, we have two numbers where the function changes its sign
so the must be (r) where $h(r) = 0$

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Question 6

Sample: 6A
Score: 8

The student earned 8 points: 2 points in part (a), 3 points in part (b), 1 point in part (c), and 2 points in part (d). The student presents correct work in parts (a), (b), and (d). In part (c) the student earned the first point for considering $g''(x)$. The student makes an error in determining $g''(x)$, and so the second point was not earned. Very few students earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 2 points in part (a), 3 points in part (b), 1 point in part (c), and no points in part (d). The student presents correct work in parts (a) and (b). In part (c) the student correctly finds $g''(x)$ and earned the first point. The second point was not earned since the student concludes that $g''(x)$ does not equal 0. In part (d) the student does not have the correct value for $h(5)$, so the first point was not earned. Since 0 is not between the student's values of $h(2)$ and $h(5)$, the student was not eligible for the second point.

Sample: 6C
Score: 3

The student earned 3 points: 2 points in part (a), no points in part (b), no points in part (c), and 1 point in part (d). Correct work is presented in part (a). In part (b) the student writes about the function g and not g'. In part (c) the student does not refer to g''. In part (d) 1 point was earned for $h(2)$ and $h(5)$. The student appeals to Rolle's Theorem instead of the Intermediate Value Theorem, and so the second point was not earned.