APCD CALCULUS: COMPARING GROWTH MODELS
Worksheet to Accompany Exploration

Directions: After completing the Exploration, answer the following additional questions. The Exploration is needed to answer some of these questions.

1. The graph of P in Model I looks like it might be a portion of a parabola with a y-intercept of 50. Such a parabola would have the equation $P(t) = kt^2 + 50$, where k is a constant. Since $\frac{dP}{dt} = 0.7P$, explain why the graph of P for Model I could not be a parabola.

2. a) The graph of P in Model I also looks like it might be a portion of an exponential curve, which would have an equation of the form $P(t) = Ce^{kt}$. Use $P(0) = 50$ and $\frac{dP}{dt} = 0.7P$ to determine the values of C and k.

   b) Use your formula from part (a) to find $\frac{dP}{dt}$ in terms of t.

c) Use your result from part (b) to verify that $\frac{dP}{dt} = 0.7P$.

d) Find $\frac{d^2P}{dt^2}$ in terms of t.

e) Use $\frac{d^2P}{dt^2}$ to explain why the graph of P is concave up for $t > 0$.

f) Find $\lim_{t \to \infty} P(t)$.

3. a) The graph of P in Model II looks like an exponential curve that has been reflected and translated. Such a curve would have an equation of the form $P(t) = 800 - Ce^{kt}$. Use $P(0) = 50$ and $\frac{dP}{dt} = 0.5(800 - P)$ to determine the values of C and k.

b) Use your formula from part (a) to find $\frac{dP}{dt}$ in terms of t.

c) Use your result from part (b) to verify that $\frac{dP}{dt} = 0.5(800 - P)$.

d) Find $\frac{d^2P}{dt^2}$ in terms of t.

e) Use $\frac{d^2P}{dt^2}$ to explain why the graph of P is concave down for $t > 0$.

f) Find $\lim_{t \to \infty} P(t)$.

4. (a) The graph of P in Model III looks like a logistic curve, which would have an equation of the form $P(t) = \dfrac{C}{1 + de^{-k \cdot C \cdot t}}$. Verify that $P(0) = 50$ when $C = 800$, $d = 15$, and $k = 0.002$.

(b) Use the formula from part (a) to find $\dfrac{dP}{dt}$ in terms of t.

(c) Use your result from part (b) to verify that $\dfrac{dP}{dt} = 0.002\,P(800 - P)$.

(d) Use the Model III graphs of $\dfrac{dP}{dt}$ and P to estimate values of P and t for the point of inflection of the graph of P.

(e) Find $\lim_{t \to \infty} P(t)$.

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TEACHER'S NOTES FOR WORKSHEET

Time of year to use: Students should already know how to find the derivatives of exponential functions. This exploration may be used as a tool for introducing logistic curves or may be used following study of logistic curves. It is not necessary to solve separable differential equations for this worksheet.

How to use in a one computer classroom: Go through the exploration with the students, allowing them to ask questions and suggest ideas for exploring. Discuss all the questions presented in the exploration itself. Then continue with the questions on the worksheet, many of which can be done without referring back to the exploration.

How to use in a computer lab setting: Allow students to use the exploration and complete the worksheet independently.

Answers:

1. If $P(t) = kt^2 + 50$, then $P'(t) = 2kt \neq 0.7P$.

2. (a) $C = 50$ and $k = 0.7$
   (b) $\frac{dP}{dt} = 35e^{0.7t}$
   (c) $\frac{dP}{dt} = 35e^{0.7t} = 0.7\left(50e^{0.7t}\right) = 0.7P$
   (d) $\frac{d^2P}{dt^2} = 0.49\left(50e^{0.7t}\right)$
   (e) $\frac{d^2P}{dt^2} > 0$ for all t, so the graph of P is concave up.
   (f) $\lim_{t\to\infty} P(t) = \infty$

3. (a) $C = 750$ and $k = -0.5$
   (b) $\frac{dP}{dt} = 375e^{-0.5t}$
   (c) $\frac{dP}{dt} = 375e^{-0.5t} = 0.5\left(750e^{-0.5t}\right) = 0.5(800 - P)$
   (d) $\frac{d^2P}{dt^2} = -0.25\left(750e^{-0.5t}\right)$
   (e) $\frac{d^2P}{dt^2} < 0$ for all t, so the graph of P is concave down.
   (f) $\lim_{t\to\infty} P(t) = 800$

4\. (a) $P(0)=\dfrac{C}{1+de^{-k\cdot C\cdot 0}}=\dfrac{800}{1+15\cdot e^{(-1.6)\cdot 0}}=\dfrac{800}{1+15}=50$

(b) $\dfrac{dP}{dt} = \dfrac{19200e^{-1.6t}}{\left(1 + 15e^{-1.6t}\right)^2}$

(c) $\dfrac{dP}{dt}=\dfrac{19200e^{-1.6t}}{\left(1+15e^{-1.6t}\right)^2}=\left(\dfrac{1.6}{1+15e^{-1.6t}}\right)\left(\dfrac{12000e^{-1.6t}}{1+15e^{-1.6t}}\right)$

$=0.002P\left(\dfrac{800+12000e^{-1.6t}}{1+15e^{-1.6t}}-\dfrac{800}{1+15e^{-1.6t}}\right)=0.002P(800-P)$

(d) $P=400$ and $t=\dfrac{\ln 15}{1.6}=1.693$

(e) $\lim_{t\to\infty}P(t)=800$