APCD CALCULUS: RELATED RATES
Worksheet to Accompany Exploration, Part 1

Directions: After completing the Exploration, answer the following additional questions. The Exploration is needed to answer some of these questions.

1. "Fill" the tanks to a "Maximum Depth" of 9. Then answer the following questions.
   a) What is the ratio of the volumes of tanks *A* and *B*?

   b) What is the ratio of the times it takes to fill the two tanks in part (a)?

   c) What is the ratio of the volumes of tanks *B* and *C*?

   d) What is the ratio of the times it takes to fill the two tanks in part (c)?

   e) What is the ratio of the volumes of tanks *C* and *D*?

   f) What is the ratio of the times it takes to fill the two tanks in part (e)?

2. Now "Fill" the tanks to a "Maximum Depth" of 6. Then repeat parts (a) – (f) in problem 1.

3. Explain why the corresponding ratios in problem 2 differ from those in problem 1.

4. Consider a tank that is a circular cylinder with a diameter of 8 feet.
   a) What is the volume if the tank is filled to a depth of 9 feet?

   b) How does the time required to fill this cylinder compare with the four tanks in the Exploration?

   c) How long would it take to fill it to a depth of 9 feet if $\frac{dV}{dt} = 20 \text{ ft}^3/\text{min}$?

   d) What is $\frac{dh}{dt}$ for this tank?

   e) Why is $\frac{dh}{dt}$ for the cylinder the same as $\frac{dh}{dt}$ for all the tanks when $h = 9$ feet?

5. Set up an integral for the volume of each tank in the Exploration. Your integral should be written in terms of dy.

6. Verify the volumes given in the Exploration by evaluating your integrals in problem 5, either analytically or using a calculator.

7. Complete the following for each tank. [Part (a) is the same for the four tanks.]

   a) Write a Chain Rule formula for $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$.

   b) Use the Fundamental Theorem of Calculus and your integrals from problem 5 to find a formula for $\frac{dV}{dh}$ for each tank.

   c) Find $\frac{dh}{dt}$ when $h = 6$, using your answers from parts (a) and (b).

   d) Compare your answers from part (c) with the values given in the Exploration.

APCD CALCULUS: RELATED RATES
Worksheet to Accompany Exploration, Part 2

Directions: After completing the Exploration, answer the following additional questions.

You have been asked to design another oil tank. The shape of the tank will be obtained by revolving the graph of $y = kx^3$ about the *y*-axis, the depth of the tank will be 9 feet, and the diameter at the top of the tank will be 8 feet.

1. Determine the value of *k* for this tank.

2. Sketch a cross sectional view of the tank, similar to those in the Exploration.

3. Set up an integral for the volume of oil when the tank the tank is filled to a depth of *h* feet. What is the volume of oil when the tank is filled to a depth of 9 feet? 6 feet?

4. Use the Fundamental Theorem of Calculus and your integral from problem 2 to find $\frac{dV}{dh}$ for this tank.

5. If $\frac{dV}{dt} = 20 \text{ ft}^3/\text{min}$, find $\frac{dh}{dt}$ when $h = 9$ and when $h = 6$.

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Worksheet to Accompany Exploration, Part 1

TEACHER'S NOTES FOR WORKSHEET

Time of year to use: Students should already know how to find the volumes of solids of revolution. This Exploration can be used as a tool for introducing related rates or following the study of related rates for review.

How to use in a one computer classroom: The class can be divided up into four groups and each group can be assigned one of the tanks for questions 6 – 8. Then the groups can share their results with the class.

How to use in a computer lab setting: Students can be encouraged to work through the worksheet on their own.

Answers:

1. a) 1.3:1 b) 1.3:1 c) 1.5:1 d) 1.5:1 e) 1.7:1 f) 1.7:1
   (Note: the difference between the ratios for volume and time for each tank are due to rounding of the reported values in the Exploration.)

2. a) 1.6:1 b) 1.6:1 c) 2.3:1 d) 2.3:1 e) 3.7:1 f) 3.7:1
   (Note: the difference between the ratios for volume and time for each tank are due to rounding of the reported values in the Exploration.)

3. Since the cross sections of the tanks are increasing at different rates, the ratios requested vary at different depths of the tanks.

4. a) 452.4 ft^3
   b) It would take the longest to fill.
   c) 22.6 minutes
   d) $\frac{dh}{dt} = \frac{\frac{dV}{dt}}{16\pi} = 0.4$ ft/min
   e) $\frac{dh}{dt}$ depends only on the cross-sectional area, which is 16π for all tanks when $h = 9$ feet. See the answer to the third question on the Exploration.

5. For each tank, $V = \int_0^9 \pi x^2 dy$

   Tank A: $V = \int_0^9 \pi \left(\frac{16}{3}\sqrt{y} \right) dy = 96\pi = 301.593$

   Tank B: $V = \int_0^9 \pi \left(\frac{16}{9} y \right) dy = 72\pi = 226.195$

Tank C: $V = \int_0^9 \pi\left(\frac{16}{81}y^2\right)dy = 48\pi = 150.796$

Tank D: $V = \int_0^9 \pi\left(\frac{16}{6561}y^4\right)dy = \frac{144\pi}{5} = 90.478$

6. See answers to #5.

7. a) $\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$

b) Tank A: $\frac{dV}{dh} = \frac{d}{dh}\int_0^h \pi\left(\frac{16}{3}\sqrt{y}\right)dy = \frac{16\pi}{3}\sqrt{h}$

Tank B: $\frac{dV}{dh} = \frac{16\pi}{9}h$

Tank C: $\frac{dV}{dh} = \frac{16\pi}{81}h^2$

Tank D: $\frac{dV}{dh} = \frac{16\pi}{6561}h^4$

c) Tank A: $\frac{dh}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dh}} = \frac{20}{\frac{16\pi}{3}\sqrt{h}}$

$$\left.\frac{dh}{dt}\right|_{h=6} = \frac{20}{\frac{16\pi}{3}\sqrt{6}} = \frac{15}{16\pi\sqrt{6}} = 0.487$$

Tank B: $\left.\frac{dh}{dt}\right|_{h=6} = \frac{20}{\frac{16\pi}{9}(6)} = 0.597$

Tank C: $\left.\frac{dh}{dt}\right|_{h=6} = \frac{20}{\frac{16\pi}{81}(6)^2} = 0.895$

Tank D: $\left.\frac{dh}{dt}\right|_{h=6} = \frac{20}{\frac{16\pi}{6561}(6)^4} = 2.014$

d) Answers are approximately equal. The slight discrepancies between the computed values and the displayed values are due to the discrete algorithm used to generate the simulations.

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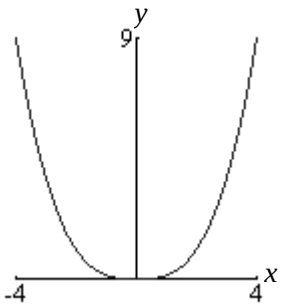
Worksheet to Accompany Exploration, Part 2

TEACHER'S NOTES FOR WORKSHEET

Time of year to use: This extension can be used as an assignment following the worksheet accompanying the Exploration. Or this extension can serve as a review of the topic "Related Rates." Students should already know how to find the volumes of solids of revolution and find related rates.

Answers:

1. $k = \frac{9}{64}$

2. 

$$y = \frac{9}{64}x^3$$

3. $V(h) = \int_0^h \pi\left(\frac{16}{\sqrt[3]{81}}\sqrt[3]{y^2}\right)dy$

$V(9) = \int_0^9 \pi\left(\frac{16}{\sqrt[3]{81}}\sqrt[3]{y^2}\right)dy = \frac{432\pi}{5} = 271.434$; $V(6) = \int_0^6 \pi\left(\frac{16}{\sqrt[3]{81}}\sqrt[3]{y^2}\right)dy = 138.095$

4. $\frac{dV}{dh} = \frac{d}{dh}\int_0^h \pi\left(\frac{16}{\sqrt[3]{81}}\sqrt[3]{y^2}\right)dy = \frac{16\pi}{\sqrt[3]{81}}\sqrt[3]{h^2}$

5. $\frac{dh}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dh}} = \frac{20}{\frac{16\pi}{\sqrt[3]{81}}\sqrt[3]{h^2}}$

$\left.\frac{dh}{dt}\right|_{h=9} = \frac{20}{\frac{16\pi}{\sqrt[3]{81}}\sqrt[3]{81}} = \frac{20}{16\pi} = 0.397 \approx 0.4$; $\left.\frac{dh}{dt}\right|_{h=6} = \frac{20}{\frac{16\pi}{\sqrt[3]{81}}\sqrt[3]{36}} = 0.521$