APCD CALCULUS: DEFINITE INTEGRAL APPROXIMATIONS
Worksheet to Accompany Exploration

Directions: After completing the Exploration, answer the following additional questions. The Exploration is needed to answer some of these questions.

1. On the first screen of the Exploration, use the slider in the lower right-hand corner to change the value of n from 2 to 5 to 10 to 20 to 50. Then answer the following questions.

   (a) Complete the table below with the values of the four integral approximations for each of the indicated values of n. Verify that the average (arithmetic mean) of the Left Riemann Sum Approximation and the Right Riemann Sum Approximation equals the value of the Trapezoidal Approximation for the values of n listed in the table.

n	Left Riemann Sum Approximation	Right Riemann Sum Approximation	Midpoint Riemann Sum Approximation	Trapezoidal Approximation
2				
5				
10				
20				
50				

   (b) Which of the four methods appears to give the best approximation of the region bounded by the curve for these values of n?

   (c) Which of the four approximations appears to be the most accurate (i.e., has the smallest absolute error) for these values of n?

   (d) Were your answers to parts (b) and (c) the same? Explain why this is or is not the case.

2. Consider the Left Riemann Sum Approximation.
   (a) Explain why the error is negative for $n > 0$.

   (b) Draw the graph of a function for which the Left Riemann Sum Approximation would have a positive error.

3. Compare the Left Riemann Sum and the Right Riemann Sum Approximations for several values of n.
   (a) Which of these has the greater absolute error for all values of n?

   (b) Draw the graph of a function for which your answer to part (a) would be changed.

   (c) Draw the graph of a function for which both the Left and Right Riemann Sum Approximations would have equal absolute errors.

4. Consider the Trapezoidal Approximation.
   (a) Explain why the error is positive for $n > 0$.

   (b) Draw the graph of a function for which the Trapezoidal Approximation would always have a negative error.

   (c) Draw a graph for which the Trapezoidal Approximation error equals 0.

   (d) How can the error for the Trapezoidal Approximation be computed from the errors for the Left and Right Riemann Sum Approximations?

5. In the Exploration, you saw that the Trapezoidal Approximation is the average (arithmetic mean) of the Left Riemann Sum Approximation and the Right Riemann Sum Approximation. (See the answer to the second question in the Exploration for a graphic interpretation and proof of this result.) Now consider the Midpoint Riemann Sum Approximation.
   (a) Let $n = 5$ in the Exploration and copy the Midpoint Riemann Sum Approximation graph below.

   (b) Now let $n = 10$ in the Exploration and look at the Left and Right Riemann Sum Approximations. Using some of the rectangles from each approximation, make a sketch that is equivalent to the Midpoint Riemann Sum Approximation with $n = 5$ that you copied in part (a). Sketch the appropriate rectangles on your drawing and label them "LR" or "RR" to indicate whether they correspond to a rectangle from the Left Riemann Sum Approximation or the Right Riemann Sum Approximation.

   (c) What relationship do you think exists between the Midpoint Riemann Sum Approximation and the Left and Right Riemann Sum Approximations?

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TEACHER'S NOTES FOR WORKSHEET

Time of year to use: This exploration can be used anytime after students have been introduced to rectangle approximation methods for the area under a curve.

How to use in a one-computer classroom: Use the worksheet as a guide for classroom discussion after discussing the Exploration. The questions asking for a drawing can be done by the students at a later time.

How to use in a computer lab setting: Allow the students to proceed on their own through the Exploration and then the worksheet.

Answers:

1. (a)

n	Left Riemann Sum Approximation	Right Riemann Sum Approximation	Midpoint Riemann Sum Approximation	Trapezoidal Approximation
2	150	450	175	300
5	170	290	210	230
10	190	250	215	220
20	202.5	232.5	216.25	217.5
50	210.8	222.8	216.6	216.8

(b) Trapezoidal Approximation or Midpoint Riemann Sum
(c) Midpoint Riemann Sum
(d) If "Yes," the smallest error would correspond to filling in the region best. If "No," then visually, the Trapezoidal Approximation looks better when $n > 10$.

2. (a) The total underestimated error is greater than the total overestimated error. This is due to a combination of two factors.
   - The absolute error is larger on intervals where the curve is steeper, and the absolute error is smaller on intervals where the curve is less steep.
   - On the intervals where f is increasing, the LRS error is negative; on the intervals where f is decreasing, the LRS error is positive.

   For this function, the graph tends to be steeper on the intervals where the graph is increasing than on the intervals where the graph is decreasing. The function is increasing on half of the interval and decreasing on half of the interval. The combination of these two factors results in negative error for the LRS Approximation.

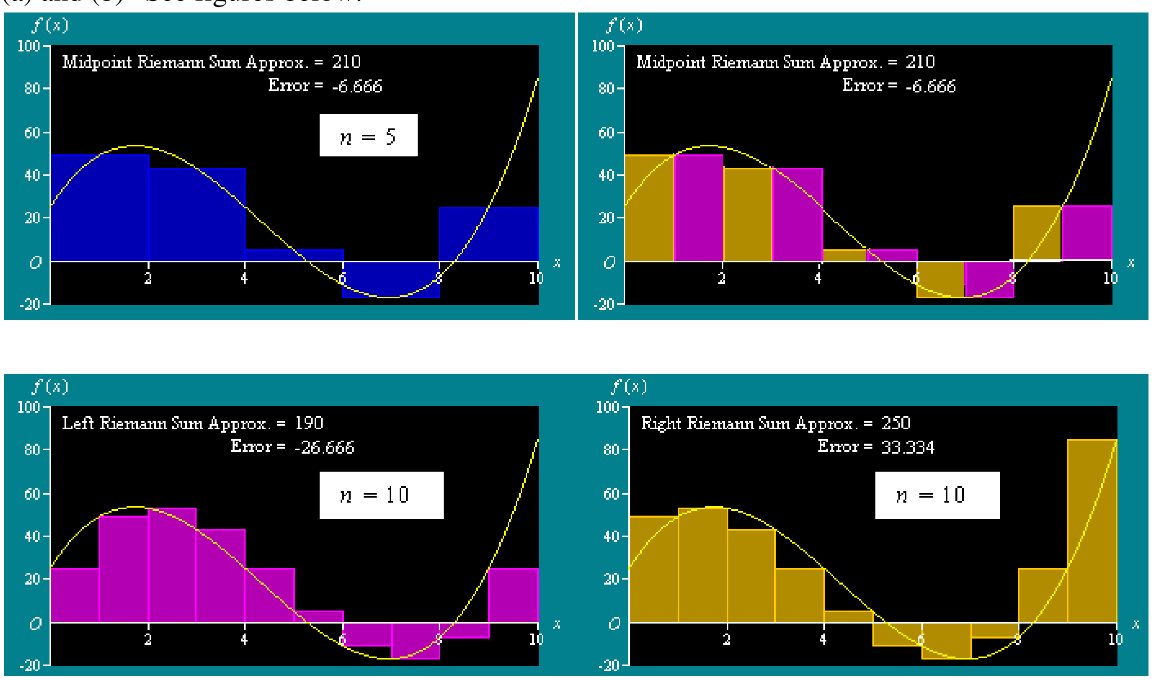
   (b) Any graph where the function is decreasing, or where the excess areas are larger than the omitted areas.

3. (a) RRS
   (b) Answers will vary. A possible answer is a graph that is decreasing and concave down, such as $f(x) = 16 - x^2$ on the closed interval $[0,4]$.
   (c) Answers will vary. Possible answers include any graph that is symmetric over the interval, such as $f(x) = \sin x$ on $[0, \pi]$ or $f(x) = x^2$ on $[-2, 2]$.

4. (a) The absolute error for the RRS is greater than the absolute error for LRS.
   OR
   The excess area on $[8.5, 10]$ is greater than the omitted area on $[0, 8.5]$.
   (b) Answers will vary. A possible answer is a graph that is above the *x*-axis and concave down, such as $f(x) = 16 - x^2$ on the closed interval $[0,4]$.
   (c) Answers will vary. Possible answers include any graph made of line segments or a graph where changes in concavity allow excess area and omitted area to cancel out, such as $f(x) = \sin x + 2$ on $[0, 2\pi]$.
   (d) Average the errors for LRS and RRS. Symbolically, we can represent this as

$$e(\text{Trap}_n) = \left[\frac{e(LRS_n) + e(RRS_n)}{2}\right]$$

5. (a) and (b) See figures below.



   (c) The 1st, 3rd, etc. RRS rectangles and the 2nd, 4th, etc. LRS rectangles with $2n$ subintervals can be arranged to correspond to the Midpoint Approximation with n intervals.